

# Thouless formula for random non-Hermitian Jacobi matrices

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## Abstract

Random non-Hermitian Jacobi matrices  $J_n$  of increasing dimension  $n$  are considered. We prove that the normalized eigenvalue counting measure of  $J_n$  converges weakly to a limiting measure  $\mu$  as  $n \rightarrow \infty$ . We also extend to the non-Hermitian case the Thouless formula relating  $\mu$  and the Lyapunov exponent of the second-order difference equation associated with the sequence  $J_n$ . The measure  $\mu$  is shown to be log-Hölder continuous.

## 1 Introduction

Let  $a_j$ ,  $b_j$ , and  $c_j$  are three given sequences of complex numbers. Consider the second-order difference equation for  $f$

$$a_j f_{j-1} + b_j f_j + c_j f_{j+1} = z f_j, \quad j = 1, 2, \dots \quad (1.1)$$

This equation can be also written as

$$\begin{pmatrix} f_{j+1} \\ f_j \end{pmatrix} = g_j \begin{pmatrix} f_j \\ f_{j-1} \end{pmatrix}, \quad j = 1, 2, \dots, \quad \text{where } g_j = \begin{pmatrix} \frac{z-b_j}{c_j} & \frac{-a_j}{c_j} \\ 1 & 0 \end{pmatrix}. \quad (1.2)$$

Denote by  $f_j(z)$  the solution of (1.1) satisfying the initial condition  $f_0 = 0$ ,  $f_1 = 1$ . In terms of the transfer matrix  $S_n(z) = g_n \cdot \dots \cdot g_1$ ,

$$\begin{pmatrix} f_{n+1}(z) \\ f_n(z) \end{pmatrix} = S_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (1.3)$$

Obviously,  $f_{n+1}(z)$  is a polynomial in  $z$  of degree  $n$ ,

$$f_{n+1}(z) = k_n \prod_{l=1}^n (z - z_l), \quad k_n = \prod_{j=1}^n 1/c_j. \quad (1.4)$$

Its roots  $z_1, \dots, z_n$  are the eigenvalues of the tridiagonal (Jacobi) matrix

$$J_n = \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{pmatrix}. \quad (1.5)$$

In this paper we are concerned with the limiting distribution of the eigenvalues of  $J_n$  as  $n \rightarrow \infty$  for random  $a_j$ ,  $b_j$ , and  $c_j$ .

If all  $b_j$  are real and  $c_j = a_{j+1}^*$  for all  $j$  the matrices  $J_n$  are Hermitian. The eigenvalue distribution of such matrices was studied extensively in the past in the context of the Anderson model, see e.g. [18, 3]. In this case the eigenvalues are always real and there are several ways to prove that the normalized eigenvalue counting measure of  $J_n$  converges to a limiting measure as  $n \rightarrow \infty$ . None of these proofs works in the non-Hermitian case and little is known about the limiting eigenvalue distribution of random non-Hermitian Jacobi matrices, however, see [5, 11].

Our interest to such matrices is partly motivated by non-Hermitian quantum mechanics of Hatano and Nelson [9, 10] which, in one dimension, leads to equation (1.1) with the coefficients  $a_j$ ,  $b_j$ , and  $c_j$  chosen randomly from the special class defined by the restrictions

$$b_j \in \mathbb{R} \text{ and } a_{j+1}^*/c_j > 0 \text{ for all } j. \quad (1.6)$$

In this class the Liouville substitution<sup>1</sup> reduces equation (1.1) to the symmetric equation

$$s_{j-1}^* \psi_{j-1} + b_j \psi_j + s_j \psi_{j+1} = z \psi_j \quad (1.7)$$

where  $s_j = c_j(a_{j+1}^*/c_j)^{1/2}$ . However, the situation here is much richer than in the Hermitian case as the choice of boundary conditions to accompany equation (1.1) has a profound effect on the spectrum of the associated Jacobi matrix. If the Dirichlet boundary conditions,  $f_0 = 0$  and  $f_{n+1} = 0$ , are chosen then the corresponding Jacobi matrix is  $J_n$  (1.5). As the Dirichlet boundary conditions are preserved by the Liouville transformation, the spectrum of  $J_n$  is real provided the coefficients  $(a_j, b_j, c_j)$  belong to the Hatano-Nelson class (1.6). On the other hand, if one imposes the periodic boundary conditions,  $f_0 = f_n$  and  $f_1 = f_{n+1}$ , then the spectrum of the corresponding Jacobi matrix turns out to be complex. This is not surprising of course as the Liouville substitution transforms the periodic boundary conditions for  $f$  into highly asymmetric boundary conditions for  $\psi$ .

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<sup>1</sup> $f_j = \theta_j \psi_j$ , where  $\theta_1 = 1$  and  $\theta_k = (\prod_{j=1}^{k-1} a_{j+1}^*/c_j)^{1/2}$  for  $k \geq 2$ .

What is surprising however is that in the limit  $n \rightarrow \infty$  the complex eigenvalues lie on analytic curves [15] and are regularly spaced even if the coefficients in equation (1.1) are chosen randomly [16]. These effects are specific to the Hatano-Nelson class and the proofs and analysis of the limiting eigenvalue distribution given in [15, 16] exploit the relation between equations (1.1) and (1.7). Of course, in the general case of arbitrary coefficients no such relation exists and one requires a different approach in order to investigate the eigenvalue distribution of  $J_n$ . We develop such an approach in the present paper.

Throughout this paper we assume that:

- A1**  $\{(a_j, b_j, c_j)\}_{j=1}^\infty$  is a sequence i.i.d. random vectors.
- A2** For some  $\delta > 0$   $E[|a_j|^\delta + |a_j|^{-\delta} + |b_j|^\delta + |c_j|^\delta + |c_j|^{-\delta}] < \infty$ .
- A3** The support of the probability distribution of the random vector  $(a_1, b_1, c_1)$  contains at least two different points  $(a, b, c)$  and  $(a', b', c')$ .

If all mass of the probability distribution of  $(a_j, b_j, c_j)$  is concentrated at one point  $(a, b, c)$  then of course we have a tridiagonal matrix with constant diagonals. This is a particular case of Töplitz matrices. Eigenvalue distribution of non-Hermitian Töplitz matrices was extensively studied in the past, see e.g. survey [21].

Our main result expresses the limiting distribution of the eigenvalues of  $J_n$  in terms of the (upper) Lyapunov exponent

$$\gamma(z) = \lim_{n \rightarrow \infty} \frac{1}{2n} \log[|f_{n+1}(z)|^2 + |f_n(z)|^2]$$

of equation (1.1). It is well known that (for every complex  $z$ ) the above limit exists with probability one and is nonrandom. This follows from Oseledec's multiplicative ergodic theorem [17]. A more subtle fact is that in our case  $\gamma(z)$  can be calculated using the well known Furstenberg formula [7], and moreover

$$\gamma(z) = \lim_{n \rightarrow \infty} \frac{1}{n} E \log \|S_n(z)\|, \quad z \in \mathbb{C}. \quad (1.8)$$

The function  $\gamma(z)$  is subharmonic in the entire complex plane [4] and bounded from below,

$$\gamma(z) \geq \frac{1}{2} E \log |a_1/c_1| \quad \text{for all } z. \quad (1.9)$$

This inequality easily follows from  $\det S_n(z) = \prod_{j=1}^n a_j/c_j$ . The subharmonicity implies that  $\Delta\gamma$ , where  $\Delta$  is the distributional Laplacian in variables  $\text{Re } z$  and  $\text{Im } z$ , defines a measure on  $\mathbb{C}$ , see e.g. [12]. Our main result is as follows.

**Theorem 1.1** *Let  $\mu_n$  be the normalized eigenvalue counting measure of  $J_n$ , i.e.  $\mu_n = \frac{1}{n} \sum_{l=1}^n \delta_{z_l}$ , where  $z_1, \dots, z_n$  are the eigenvalues of  $J_n$ . Then:*

- (a) *With probability one,  $\mu_n$  converges weakly to  $\mu = \frac{1}{2\pi} \Delta\gamma$  as  $n \rightarrow \infty$ .*

(b) (Thouless formula) For every  $z \in \mathbb{C}$

$$\gamma(z) = \int_{\mathbb{C}} \log |w - z| d\mu(w) - E \log |c_1|. \quad (1.10)$$

(c) The limiting eigenvalue counting measure  $\mu$  is log-Hölder continuous. More precisely, for any  $B_{z_0, \delta} = \{z : |z - z_0| \leq \delta\}$ ,  $0 < \delta < 1$ ,

$$\mu(B_{z_0, \delta}) \leq \frac{C(z_0, \delta)}{\log \frac{1}{\delta}}, \quad (1.11)$$

where  $C(z_0, \delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

We deduce Theorem 1.1 from Theorem 1.2 which is of independent interest in the context of second order difference equations.

**Theorem 1.2** *With probability one*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |f_{n+1}(z)| = \gamma(z) \quad (1.12)$$

for almost all  $z$  with respect to the Lebesgue measure on  $\mathbb{C}$ .

To prove Theorem 1.2, we use the theory of products of random matrices. Of course this is unnecessary in the Hermitian case. In this case (1.12) and the Thouless formula (1.10) follow directly from the fact that  $\mu_n$  converges weakly to a limiting measure  $\mu$  [1, 6] and the latter can be established independently and by more elementary means. We would like to emphasize that in the non-Hermitian case we follow the opposite direction route: the weak convergence of  $\mu_n$  and the Thouless formula are deduced from (1.12). To this end we make use of the relation

$$\mu_n = \frac{1}{2\pi n} \Delta \log |f_{n+1}|, \quad (1.13)$$

where the equality is to be understood in the sense of distribution theory. Relation (1.13) is well known in the function theory. It holds for arbitrary polynomial of degree  $n$  and can be easily derived with the help of the Gauss-Green formula. In this general setup it was shown by Widom [20, 21] that if the measures  $\mu_n$  for all  $n$  are supported inside a bounded region and in the limit  $n \rightarrow \infty$  the function  $p_n(z) = \int_{\mathbb{C}} \log |z - w| d\mu_n(w)$  converges to a limiting function  $p(z)$  almost everywhere in the complex plane then  $\mu_n$  converges weakly to  $\mu = \frac{1}{2\pi} \Delta p$ . We shall need the following simple extension of this result to the case when the supports of  $\mu_n$  are not necessarily bounded.

Let  $A_n$  be a (deterministic) sequence of square matrices of increasing dimension  $n$ , and

$$p_n(z) = \frac{1}{n} \log |\det(A_n - zI_n)| = \int_{\mathbb{C}} \log |w - z| d\mu_n(w),$$

where  $I_n$  is  $n \times n$  identity matrix and  $\mu_n = \frac{1}{2\pi} \Delta p_n$  is the normalized eigenvalue counting measure of  $A_n$ . Define

$$\tau_R = \limsup_{n \rightarrow \infty} \int_{|w| \geq R} \log |w| d\mu_n(w), \quad R \geq 1. \quad (1.14)$$

**Proposition 1.3** *Assume that there is a function  $p: \mathbb{C} \rightarrow [-\infty, +\infty)$  such that  $p_n(z) \rightarrow p(z)$  as  $n \rightarrow \infty$  almost everywhere in  $\mathbb{C}$ . If  $\tau_1 < +\infty$  then it follows that  $p$  is locally integrable,  $\mu = \frac{1}{2\pi} \Delta p$  is a unit mass measure,*

$$\int_{|w| \geq 1} \log |w| d\mu(w) \leq \tau_1 < +\infty, \quad (1.15)$$

*and the sequence of measures  $\mu_n$  converges weakly to  $\mu$  as  $n \rightarrow \infty$ . If, in addition,  $\lim_{R \rightarrow \infty} \tau_R = 0$  then we also have that*

$$p(z) = \int_{\mathbb{C}} \log |w - z| d\mu(w). \quad (1.16)$$

*Remark.* In view of (1.15), the integral on the RHS in (1.16) is a locally integrable function of  $z$  taking values in  $[-\infty, +\infty)$ .

For the sake of completeness, we give a proof of this Proposition in Appendix A.

In order to estimate the tails of eigenvalue distributions as required in the above Proposition 1.3 we use the following inequalities<sup>2</sup>:

$$\tau_1 \leq \limsup_{n \rightarrow \infty} \frac{1}{2n} \log \det(I_n + A_n A_n^*), \quad (1.17)$$

and for any  $R > 1$  and  $\delta > 0$

$$\tau_R \leq \frac{1}{\log^\delta R} \limsup_{n \rightarrow \infty} \frac{1}{2^{1+\delta} n} \operatorname{tr} \log^{1+\delta}(I_n + A_n A_n^*). \quad (1.18)$$

These inequalities can be derived with the help of Weyl's Majorant Theorem, for details of derivation see Appendix B.

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<sup>2</sup>Note that  $\log \det(I_n + A_n A_n^*) = \operatorname{tr} \log(I_n + A_n A_n^*)$ .

Let us now return to the random Jacobi matrices  $J_n$ . Straightforward but tedious calculations show<sup>3</sup> that

$$\frac{1}{n} \operatorname{tr} \log^{1+\delta}(I_n + J_n J_n^*) \leq \frac{\alpha}{n} \sum_{j=1}^n \log^{1+\delta}(1 + \beta |\mathbf{v}_j|^2), \text{ where } \mathbf{v}_j = (a_j, b_j, c_j),$$

for some  $\alpha, \beta > 0$  independent of  $\mathbf{v}_j$ 's and  $n$ . Therefore if the random sequence  $\mathbf{v}_j$  is stationary and

$$E \log^{1+\delta}[1 + |\mathbf{v}_1|^2] < \infty \quad \text{for some } \delta > 0 \quad (1.19)$$

then the Ergodic Theorem asserts that with probability one the limits in (1.17) (1.18) are finite which implies  $\tau_1 < \infty$  and  $\lim_{R \rightarrow \infty} \tau_R = 0$ , as required in Proposition 1.3. The assumptions of stationarity and (1.19) are less restrictive than assumptions A1-A3. However we are only able to prove Theorem 1.2 (which is the main ingredient to our proof of Theorem 1.1) under these more restrictive assumptions.

## 2 Products of random matrices

Our proof of Theorem 1.2 makes use of several facts from the theory of products of random  $2 \times 2$  matrices. We list these facts below (Propositions 2.1 - 2.3).

Let  $\nu$  be a probability distribution on the group  $Gl(2, \mathbb{C})$  of invertible complex  $2 \times 2$  matrices and  $g_k$  be an infinite sequence of independent samples from this distribution.

As before  $S_n = g_n \cdot \dots \cdot g_1$  for  $n = 1, 2, \dots$ . By  $P(\mathbb{C}^2)$  we denote the projective space on which every non-degenerate matrix  $g$  acts in a natural way. Let  $\kappa$  be a probability measure on  $P(\mathbb{C}^2)$ . We say that  $g$  preserves  $\kappa$  if  $\kappa(g^{-1}.B) = \kappa(B)$  for any Borel set  $B$  (here  $g.x$  is the result of the action of  $g$  on  $x \in P(\mathbb{C}^2)$ ). By  $G_\nu$  we denote the closure of the subgroup of  $Gl(2, \mathbb{C})$  generated by all matrices belonging to the support of  $\nu$ . We say that  $G_\nu$  preserves  $\kappa$  if  $\kappa$  is preserved by every  $g \in G_\nu$ .

**Proposition 2.1** *Let  $\lambda_1^{(n)} \geq \lambda_2^{(n)}$  be the singular values of  $S_n$ . If*

$$E \log \|g\| \text{ and } E \log |\det g| \text{ are both finite} \quad (2.1)$$

*then with probability one the following limits*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \lambda_j^{(n)} = \gamma_j, \quad j = 1, 2, \quad (2.2)$$

*exist and are nonrandom.*

The limiting values  $\gamma_1$  and  $\gamma_2$  are called the Lyapunov exponents of the sequence  $S_n$ .

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<sup>3</sup>For any Hermitian matrix  $H = \|H_{jk}\|_{j,k=1}^n$  we have  $H \leq D = \operatorname{diag}(d_1, \dots, d_n)$  with  $d_j = \sum_{k=1}^n |H_{jk}|$ ,  $j = 1, \dots, n$ . Therefore if  $f$  is a nondecreasing function then, by the Courant-Fisher minimax principle,  $\operatorname{tr} \log f(H) \leq \operatorname{tr} \log f(D) = \sum_{j=1}^n f(d_j)$ .

**Proposition 2.2** *If in addition to condition (2.1), no measure  $\kappa$  is preserved by  $G_\nu$  then the Lyapunov exponents of the sequence  $S_n$  are distinct, i.e.  $\gamma_1 > \gamma_2$ .*

**Proposition 2.3** *If condition (2.1) is satisfied and no measure  $\kappa$  is preserved by  $G_\nu$  then*

(i) *For any unit vector  $x$  the probability is one that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|S_n x\| = \gamma_1. \quad (2.3)$$

(ii) *If in addition  $E(\|g\|^\delta + \|g^{-1}\|^\delta) < \infty$  for some  $\delta > 0$  then for any positive  $\varepsilon$  there is a constant  $\rho(\varepsilon) > 0$  such that uniformly in  $x$ ,  $\|x\| = 1$ ,*

$$\text{Prob}(|\log \|S_n x\| - n\gamma_1| \geq \varepsilon n) \leq e^{-n\rho(\varepsilon)}. \quad (2.4)$$

*Remarks.* 1. As all norms in  $\mathbb{C}^2$  are equivalent, the choice of norm in (2.3) and (2.4) is not important. However it is convenient to deal with the standard Euclidian norm.

2. Propositions 2.1 - 2.3 are well known in the classical case of the real matrices, see e.g. [17, 2] for proofs of Propositions 2.1 and 2.3 and [7, 19] for proofs of Proposition 2.2. For complex matrices, Propositions 2.1 and 2.3 are proved in the same way as in [17, 2]. However, the proof of Proposition 2.2 is somewhat different from that given in [7, 19]. We shall now discuss the necessary changes which would allow the interested reader to reconstruct the proof in question simply by examining the one in [19]. Namely, the main ingredient of this proof is the fact that the mapping  $g \mapsto T_g$ , where

$$(T_g f)(x) = f(g^{-1}x) \|g^{-1}x\|^{-\frac{m}{2}},$$

defines a unitary representation of the group  $SL(m, \mathbb{R})$  in Hilbert space  $L_2(\mathcal{S}_m, dl)$  with  $dl$  being the natural Lebesgue measure on the unit sphere  $\mathcal{S}_m \in \mathbb{R}^m$ . (Obviously, we are interested in the case when  $m = 2$ .)

In the case of the complex space the representation is defined by

$$(T_g f)(x) = f(g^{-1}x) \|g^{-1}x\|^{-m},$$

in Hilbert space  $L_2(\mathcal{S}_m, dl)$  with  $dl$  being again the natural Lebesgue measure on the unit sphere  $\mathcal{S}_m \in \mathbb{C}^m$ . After that the proof proceeds in the way suggested in [19].

### 3 Proofs of Theorems 1.1 and 1.2

In order to be able to apply Propositions 2.1 – 2.3 we have to verify that under assumptions A1–A3 our matrices  $g_j$  defined in (1.2) satisfy the conditions of these Propositions.

Is is apparent that assumption A2 guarantees that condition (2.1) is satisfied and  $E(||g||^\delta + ||g^{-1}||^\delta) < \infty$ . It remains to check that assumption A3 implies that no measure  $\kappa$  is preserved by  $G_\nu$  (here  $\nu$  is the measure induced on the group of matrices by the distribution of  $(a_1, b_1, c_1)$ ). To this end we note that if

$$g = \begin{pmatrix} \frac{z-b}{c} & \frac{-a}{c} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad g' = \begin{pmatrix} \frac{z-b'}{c'} & \frac{-a}{c'} \\ 1 & 0 \end{pmatrix}$$

then

$$gg'^{-1} = \begin{pmatrix} \frac{c'a}{a'c} & \frac{z-b}{c} - \frac{(z-b')a}{a'c} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g'^{-1}g = \begin{pmatrix} 1 & 0 \\ \frac{z-b'}{a'} - \frac{(z-b)c'}{ca'} & \frac{c'a}{a'c} \end{pmatrix} \quad \text{and}$$

It remains to check that for almost all  $z$  that the group  $G$  generated by the matrices  $g, g'$  is rich enough in the sense that no measure is preserved by all matrices of this group. The main idea is as follows. For a "typical"  $z$  we construct two matrices, say  $B$  and  $D$ , from  $G$  such that the eigenvalues of  $B$  are of different moduli. It is easy to see then that the only measure preserved by all matrices of the form  $B^n$ ,  $-\infty < n < \infty$  is the one supported by the lines in  $P(\mathbb{C}^2)$  generated by the eigenvectors of  $B$ . The matrix  $D \in G$  is then chosen so that its action on  $P(\mathbb{C}^2)$  does not preserve these lines which means that the measure in question does not exist. We would like to emphasize that the presence of the parameter  $z$  plays a crucial role in this situation.

More precisely, if  $z$  is such that

$$2 \arg(z - b) \neq \arg(ac)$$

then the matrix  $g$  has eigenvalues with different moduli. In other words the moduli are different if  $z$  does not belong to a certain half line. The  $g'$  plays then the role of  $D$  (once again when  $z$  lies outside of certain curves). This statement can be checked by direct calculation and is sufficient for our purposes.

However, in some important cases much more precise statements can be made. In particular if  $\frac{c'a}{a'c} = 1$  then each of triangular matrices  $(gg'^{-1})$  and  $g'^{-1}g$  is non-trivial for all but may be two values of  $z$  and a similar idea applies, see [2] page 213.

Now we are in a position to apply Propositions 2.1 - 2.3. For any two non-zero vectors  $x$  and  $y$  define

$$d(x, y) = \sqrt{1 - \frac{|(x, y)|^2}{(x, x)(y, y)}},$$

where  $(\cdot, \cdot)$  is the scalar product in  $\mathbb{C}^2$ . The function  $d(x, y)$  is the natural angular distance between  $x$  and  $y$  on the projective space  $P(\mathbb{C}^2)$ .

The following Lemma is the key element in the proof of Theorem 1.2. (In this Lemma and thereafter the abbreviation a.s. refers to the probability measure, i.e. any equality with the letters a.s. above it holds with probability one)

**Lemma 3.1** *Suppose that the conditions of Propositions 2.1 - 2.3 are satisfied. If  $y_n$  is a sequence of random unit vectors in  $\mathbb{C}^2$  such that*

$$||S_n y_n|| = e^{n\gamma_2 + \epsilon_n}, \quad \text{where } \epsilon_n \stackrel{\text{a.s.}}{=} o(n) \text{ as } n \rightarrow \infty, \quad (3.1)$$



then for any fixed unit vector  $x$  and any  $\delta > 0$  there is a constant  $r(x, \delta) > 0$  such that

$$\text{Prob} \{d(x, y_n) \leq e^{-n\delta}\} \leq e^{-nr(x, \delta)} \quad (3.2)$$

for all sufficiently large  $n$ .

*Proof.* For any  $n$  one can always find two orthogonal unit vectors  $u_n$  and  $v_n$  such that  $S_n^* S_n u_n = \lambda_1^{(n)}$  and  $S_n^* S_n v_n = \lambda_2^{(n)}$ . In view of Proposition 2.1,

$$\|S_n u_n\| = e^{n\gamma_1 + \epsilon'_n} \text{ and } \|S_n v_n\| = e^{n\gamma_2 + \epsilon''_n}, \text{ where } \epsilon'_n, \epsilon''_n \stackrel{\text{a.s.}}{=} o(n).$$

Obviously the sequence  $v_n$  satisfies condition (3.1) and we first prove the large deviation estimate (3.2) for this sequence.

Let  $x$  be a fixed unit vector. Then  $x = (x, u_n)u_n + (x, v_n)v_n$  for every  $n$ , and, since  $|(x, u_n)| = d(x, v_n)$  and  $|(x, v_n)| \leq 1$ , we have that

$$\|S_n x\| \leq d(x, v_n)\|S_n u_n\| + \|S_n v_n\|.$$

Therefore if  $d(x, v_n) \leq e^{-n\delta}$  then

$$\log \|S_n x\| \leq n\gamma_1 + \log(e^{-n\delta + \epsilon'_n} + e^{-n(\gamma_1 - \gamma_2) + \epsilon''_n}),$$

and hence with probability one,

$$\log \|S_n x\| - n\gamma_1 \leq -n \min(\delta, \gamma_1 - \gamma_2) + o(n).$$

It follows now from Proposition 2.3 that

$$\text{Prob} \{d(x, v_n) \leq e^{-n\delta}\} \leq e^{-nr(x, \delta)} \quad (3.3)$$

for some  $r(x, \delta)$  and all  $n > n_0$  where  $n_0$  depends on the matrices  $S_n$ , and also on  $x$  and  $\delta$ .

Now, let  $y_n$  be an arbitrary sequence of random unit vectors satisfying condition (3.1), and let  $y_n^\perp$  be a sequence of unit vectors orthogonal to  $y_n$ , i.e.  $(y_n, y_n^\perp) = 0$  for all  $n$ . Obviously,  $d(u_n, y_n^\perp) = |(u_n, y_n)|$  and, since  $S_n^* S_n u_n = e^{2n\gamma_1 + 2\epsilon'_n} u_n$ , we have that with probability one

$$d(u_n, y_n^\perp) = e^{-2n\gamma_1 + o(n)} |(S_n u_n, S_n y_n)| \leq e^{-n(\gamma_1 - \gamma_2) + o(n)}.$$

It is then apparent that  $d(v_n, y_n)$  is also exponentially small for large  $n$  and therefore the large deviation estimate (3.2) for  $y_n$  follows from (3.3).  $\square$

*Proof of Theorem 1.2.* Let

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad y_n = \begin{pmatrix} f_{n+1}(z) \\ f_n(z) \end{pmatrix}, \quad n = 1, 2, \dots$$

Then

$$d^2(x, y_n) = \frac{|f_{n+1}(z)|^2}{|f_{n+1}(z)|^2 + |f_n(z)|^2} = \frac{|f_{n+1}(z)|^2}{||y_n||^2},$$

and therefore

$$\frac{1}{n} \log |f_{n+1}(z)| = \frac{1}{n} \log d(x, y_n) + \frac{1}{n} \log ||y_n||. \quad (3.4)$$

In view of (1.3) and Proposition 2.3(i),

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log ||y_n|| \stackrel{\text{a.s.}}{=} \gamma_1(z), \quad (3.5)$$

where  $\gamma(z)$  is the upper Lyapunov exponent of the sequence of transfer matrices  $S_n(z)$ . On the other hand,  $S_n^{-1}(z)y_n = (1, 0)^T$ , and therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{||S_n^{-1}(z)y_n||}{||y_n||} \stackrel{\text{a.s.}}{=} -\gamma_1(z).$$

It follows now from Lemma 3.1 (applied to the matrices  $S_n^{-1}(z)$  and the vectors  $x$  and  $y_n/||y_n||^4$ ) and the Borel-Cantelli Lemma that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log d(x, y_n) \stackrel{\text{a.s.}}{=} 0.$$

Therefore, in view of (3.4) and (3.5), for any fixed  $z$  the probability is one that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |f_{n+1}(z)| = \gamma_1(z). \quad (3.6)$$

But then the probability is one that (3.6) holds almost everywhere in the complex plane. This follows from the Fubini Theorem. Our proof of Theorem 1.2 is complete.

*Proof of Theorem 1.1.* As explained in introduction, under assumptions A1 – A3, the probability is one that  $\tau_1 \leq C$  for some non-random  $C < +\infty$  and  $\lim_{R \rightarrow \infty} \tau_R = 0$ . Therefore parts (a) and (b) of Theorem 1.1 follow immediately from Theorem 1.2 by the way of Proposition 1.3.

The log-Hölder continuity of  $\mu$  is a corollary of the Thouless formula and the fact that the Lyapunov exponent  $\gamma(z)$  is bounded from below. This is very much in the same way as in the Hermitian case, see [4].

To prove (1.11), we first note that the integral  $\int_{\mathbb{C}} \log |w - z| d\mu(w)$  converges absolutely for every  $z$ . Indeed, it follows from (1.15) that

$$\int_{|w-z| \geq 1} \log |w - z| d\mu(w) < +\infty,$$

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<sup>4</sup>If  $\gamma_1$  and  $\gamma_2$  are the Lyapunov exponents of a sequence  $S_n$  then the sequence  $S_n^{-1}$  has the Lyapunov exponents  $-\gamma_2$  and  $-\gamma_1$ .

and this inequality together with the Thouless formula and the lower bound (1.9) imply that

$$\int_{|w-z|\leq 1} |\log |w-z|| d\mu(w) < +\infty$$

as well. Therefore,

$$C(z, \delta) := \int_{|w-z|\leq \delta} |\log |w-z|| d\mu(w) \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (3.7)$$

Obviously, for  $\delta < 1$ ,

$$C(z, \delta) = \int_{|w-z|\leq \delta} \log(1/|w-z|) d\mu(w) \geq \frac{\mu(B_{z,\delta})}{\log(1/\delta)},$$

and part (c) of Theorem 1.1 follows. Our proof of Theorem 1.1 is now complete.

## A Appendix

*Proof of Proposition 1.3.* The local integrability of  $\log |z|$  and the condition  $\tau_1 < +\infty$  imply that the functions  $p_n(z)$  are uniformly integrable on bounded sets in  $\mathbb{C}$ . It follows from this that  $p(z)$  is locally integrable and  $p_n \rightarrow p$  as  $n \rightarrow \infty$  in  $D'(\mathbb{C})$ , the space of Schwartz distributions in  $\mathbb{C}$ . Since  $\Delta$  is continuous on distributions, we also have that  $\Delta p_n \rightarrow \Delta p$  in  $D'(\mathbb{C})$ . Obviously  $\Delta p \geq 0$ , hence  $\Delta p$  is defined by a measure, see e.g. [13]. As any sequence of measures converging as distributions must converge weakly we conclude that  $\mu_n = \frac{1}{2\pi} \Delta p_n \rightarrow \mu = \frac{1}{2\pi} \Delta p$  weakly as measures.

For any  $R > 1$ ,

$$\int_{|w|\geq R} d\mu_n(w) \leq \frac{1}{\log |R|} \int_{|w|\geq 1} \log |w| d\mu_n(w).$$

Therefore the inequality  $\tau_1 < +\infty$  implies that the sequence of measures  $\mu_n$  is tight, and hence cannot lose mass. As each of  $\mu_n$  has unit mass, so has the limiting measure  $\mu$ .

It follows from the weak convergence of  $\mu_n$  to  $\mu$  and (1.14) that

$$\int_{1\leq |w|\leq R} \log |w| d\mu(w) \leq \lim_{n\rightarrow\infty} \int_{1\leq |w|\leq 2R} \log |w| d\mu_n(w) \leq \tau_1$$

for any  $R > 1$ . This implies (1.15). Similarly, if  $\lim_{R\rightarrow\infty} \tau_R = 0$  then

$$\lim_{R\rightarrow\infty} \int_{|w|\geq R} \log |w| d\mu(w) = 0. \quad (A.1)$$

It remains to prove relation (1.16). It will suffice to show that

$$p_n \rightarrow \int_{\mathbb{C}} \log |w - \cdot| d\mu(w) \quad \text{in } D'(\mathbb{C}) \quad (A.2)$$

when  $n \rightarrow \infty$ . Let  $\psi(z)$  be a continuous function with bounded support. Then

$$\int_{\mathbb{C}} p_n(z) \psi(z) d^2 z = \int_{\mathbb{C}} g(w) d\mu_n(w)$$

with

$$g(w) = \int_{\mathbb{C}} \psi(z) \log |w - z| d^2 z.$$

The function  $g$  is continuous and  $g(w) = O(\log |w|)$  when  $|w| \rightarrow \infty$ . Assume now that  $\lim_{R \rightarrow \infty} \tau_R = 0$ . Then

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|w| \geq R} |g(w)| d\mu_n(w) = 0,$$

and

$$\lim_{R \rightarrow \infty} \int_{|w| \geq R} |g(w)| d\mu(w) = 0$$

because of (A.1). It now follows from the weak convergence of  $\mu_n$  to  $\mu$  that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{C}} g(w) d\mu_n(w) = \int_{\mathbb{C}} g(w) d\mu(w).$$

Therefore

$$\lim_{n \rightarrow \infty} \int_{\mathbb{C}} p_n(z) \psi(z) d^2 z = \int_{\mathbb{C}} g(w) d\mu(w) = \int_{\mathbb{C}} \left\{ \int_{\mathbb{C}} \log |w - z| d\mu(z) \right\} \psi(z) d^2 z,$$

and (A.2) follows.

## B Appendix

*Derivation of inequalities (1.17) and (1.18).* Let  $z_1, \dots, z_n$  and  $s_1, \dots, s_n$  be respectively the eigenvalues and singular values of  $A_n$  labeled so that  $|z_1| \geq |z_2| \geq \dots \geq |z_n|$  and  $s_1 \geq s_2 \geq \dots \geq s_n$ . Weyl's Majorant Theorem, see [14], page 39, asserts that

$$\sum_{j=1}^m F(|z_j|) \leq \sum_{j=1}^m F(s_j), \quad m = 1, 2, \dots, n,$$

for any function  $F(t)$  ( $0 \leq t < \infty$ ) such that  $F(e^x)$  is convex on  $\mathbb{R}$ . Obviously the function  $\log^{1+\delta}(t)$  satisfies this requirement for  $\delta \geq 0$ , and therefore

$$\int_{|w| \geq 1} \log^{1+\delta} |w| d\mu_n(w) = \frac{1}{n} \sum_{|z_j| \geq 1} \log^{1+\delta} |z_j| \leq \frac{1}{n} \sum_{|z_j| \geq 1} \log^{1+\delta} |z_j| \leq \frac{1}{n} \sum_{j=1}^m \log^{1+\delta} s_j$$

where  $m$  is the number of eigenvalues of  $A_n$  such that  $|z_j| \geq 1$ . Obviously,

$$\sum_{j=1}^m \log s_j = \frac{1}{2^{1+\delta}} \sum_{j=1}^m \log^{1+\delta}(s_j^2) \leq \frac{1}{2^{1+\delta}} \sum_{j=1}^n \log^{1+\delta}(1 + s_j^2) = \frac{1}{2^{1+\delta}} \operatorname{tr} \log^{1+\delta}(I_n + A_n A_n^*),$$

and therefore

$$\int_{|w| \geq 1} \log^{1+\delta} |w| d\mu_n(w) \leq \frac{1}{2^{1+\delta}} \operatorname{tr} \log^{1+\delta}(I_n + A_n A_n^*), \quad \delta \geq 0,$$

which implies (1.17) and (1.18).

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